Classical solvability of nonlinear initial-boundary problems for first-order hyperbolic systems

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Abstract

We prove the global classical solvability of initial-boundary problems for semilinear first-order hyperbolic systems subjected to local and nonlocal nonlinear boundary conditions. We also establish lower bounds for the order of nonlinearity demarkating a frontier between regular cases (classical solvability) and singular cases (blow-up of solutions).

1 Introduction

We study existence, uniqueness, and continuous dependence on initial data of classical solutions to initial-boundary problems for semilinear hyperbolic systems with nonlinear nonlocal boundary conditions. Specifically, in the domain $\Pi = \{(x,t)\,|\,0<x<1,\,t>0\}$ we address the following problem:

$$(\partial_t + \Lambda(x,t)\partial_x)u = f(x,t,u), \qquad (x,t)\in\Pi \tag{1}$$

$$u(x,0) = \varphi(x), \qquad x\in(0,1) \tag{2}$$

$$\begin{array}{lll} u_i(0,t) = h_i(t,v(t)), & k+1\le i\le n, & t\in(0,\infty) \\ u_i(1,t) = h_i(t,v(t)), & 1\le i\le k, & t\in(0,\infty) \end{array} \tag{3}$$

where u, f, and φ are real n-vectors, $\Lambda = diag(\lambda_1, \dots, \lambda_n)$ is a diagonal matrix, $\lambda_1, \dots, \lambda_k < 0$, $\lambda_{k+1}, \dots, \lambda_n > 0$ for some $1 \le k \le n$, and $v(t) = (u_1(0,t), \dots, u_k(0,t), u_{k+1}(1,t), \dots, u_n(1,t))$. Note that the system (1) is non-strictly hyperbolic and the boundary of Π is non-characteristic. We will denote $h = (h_1, \dots, h_n)$.

Special cases of (1)–(3) arise in laser dynamics (Jochmann and Recke, 1999; Radziunas et al., 2000; Sieber, Recke and Schneider, 2004) and chemical kinetics (Zelenjak, 1966; Lyul'ko, 2002).

We establish a global existence-uniqueness classical result for the problem (1)–(3). Its novelty consists in allowing nonlinear local and nonlocal boundary conditions and in allowing non-Lipschitz nonlinearities in (1) and (3). Namely, either the functions f and h can be both non-Lipschitz with $\|f\| = O(\|u\| \log^{1/4} \log \|u\|)$ and $\|g\| = O(\|v\| \log^{1/4} \log \|v\|)$ or f can be non-Lipschitz with $\|f\| = O(\|u\| \log \log \|u\|)$ while g in this case should be Lipschitz (a more detailed description is given in Section 3, see also Remark 3.2). Here and below by $\|\cdot\|$ we denote the Euclidian norm in $\mathbb{R}^n$. It turns out that, in these conditions, solutions to (1)–(3) have the same qualitative behaviour as in the linear case.

Our result should be contrasted to the known fact (see, e.g., Li Ta-tsien, 1994; Alinhac, 1995) that if the right hand side $f(x,t,u)$ of (1) has at least quadratic growth in u, then classical solutions to (1)–(3) in general fail to exist globally in time. More precisely, they blow-up in a finite time, creating singularities. Thus, we show that a "frontier" between growth rates of f ensuring regular behavior of the system and causing singular behavior lies somewhere between $\|u\| \log \log \|u\|$ and $\|u\|^2$. An intriguing problem is to make the gap closer.

For different aspects of the subject we refer the reader to Myshkis and Filimonov (1981, 2003) and to Kmit (2006a, 2006b). Myshkis and Filimonov (1981) investigate problems with nonlinear boundary conditions, but only of the *local* type. In contrast to this, our case includes nonlinear *nonlocal* boundary conditions. Furthermore, Myshkis and Filimonov (1981) make an essential assumption on nonlinearities, namely, that f and h are globally Lipshitz in, respectively, u and v. This means that f (resp., g) admits no more than linear growth in u (resp., v) as $\|u\| \to \infty$ (resp., $\|v\| \to \infty$).

Kmit (2006a, 2006b) considers the problem (1)–(3) admitting strong singularities both in the differential equations and in the initial-boundary conditions. The author proves a general existence-uniqueness result in the Colombeau algebra of generalized functions and derives asymptotic estimates

for generalized solutions.

2 The Case of Lipschitz Nonlinearities: Existence, Uniqueness, and Continuous Dependence

If the initial data of the problem (1)–(3) are sufficiently smooth, then the zero-order and the first-order compatibility conditions between (1) and (2) are given by equalities

$$\begin{array}{rclr} \varphi_i(0) & = & h_i(0, v(0)), & k+1 \le i \le n, \\ \varphi_i(1) & = & h_i(0, v(0)), & 1 \le i \le k, \end{array} \tag{4}$$

and

$$\begin{aligned} & f_i(0,0,\varphi(0)) - \lambda_i(0,0)\varphi_i'(0) = \partial_t h_i(0, v(0)) \\ & \qquad + \nabla_v h_i(0, v(0)) \cdot v'(0), \quad k+1 \le i \le n; \\ & f_i(1,0,\varphi(1)) - \lambda_i(1,0)\varphi_i'(1) = \partial_t h_i(0, v(0)) \\ & \qquad + \nabla_v h_i(0, v(0)) \cdot v'(0), \quad 1 \le i \le k, \end{aligned} \tag{5}$$

where

$$\begin{aligned} v(0) &= \Big(\varphi_1(0), \dots, \varphi_k(0), \varphi_{k+1}(1), \dots, \varphi_n(1)\Big), \\ v'(0) &= \Big(f_1(0,0,\varphi(0)) - \lambda_1(0,0)\varphi_1'(0), \dots, \\ & \quad f_k(0,0,\varphi(0)) - \lambda_k(0,0)\varphi_k'(0), \\ & \quad f_{k+1}(1,0,\varphi(1)) - \lambda_{k+1}(1,0)\varphi_{k+1}'(1), \\ & \quad \dots, f_n(1,0,\varphi(1)) - \lambda_n(1,0)\varphi_n'(1)\Big), \end{aligned}$$

and " $\cdot$ " denotes the scalar product in $\mathbb{R}^n$.

Theorem 2.1. *Assume that the initial data λ_i and f_i are continuous, λ_i and φ_i are C^1-smooth in x, f_i are C^1-smooth in x and u, h_i are C^1-smooth in both arguments. Let $\nabla_y f(x,t,y)$ be bounded on $K \times \mathbb{R}^n$ for every compact $K \subset \overline{\Pi}$ and $\nabla_z h(t,z)$ be bounded on $K \times \mathbb{R}^n$ for every compact $K \subset [0,\infty)$. If the zero-order and the first-order compatibility conditions (4) and (5) are fulfilled, then the problem (1)–(3) has a unique classical solution in Π.*

Proof. An equivalent integral-operator representation of (1)–(3) can be written in the form

$$u_i(x,t) = (R_i u)(x,t) + \int\limits_{t_i(x,t)}^{t} \Big[u(\omega_i(\tau;x,t),\tau)\cdot\int\limits_0^1 \nabla_u f_i(\omega_i(\tau;x,t),\tau,\sigma u)\,d\sigma + f_i(\omega_i(\tau;x,t),\tau,0)\Big]\,d\tau, \qquad 1\le i\le n, \tag{6}$$

where

$$(R_i u)(x,t) = \varphi_i(\omega_i(0;x,t))$$

if $t_i(x,t)=0$ and

$$(R_i u)(x,t) = = v(t_i(x,t))\cdot\int\limits_0^1 \nabla_v h_i(t_i(x,t),\sigma v)\,d\sigma + h_i(t_i(x,t),0)$$

otherwise. Here $\omega_i(\tau;x,t)$ denotes the i-th characteristic of (1) passing through $(x,t)\in\overline{\Pi}$ and $t_i(x,t)$ denotes the smallest value of $\tau\ge 0$ at which $\xi=\omega_i(\tau;x,t)$ riches $\partial\Pi$. Given $T>0$, denote

$$\Pi^T = \{(x,t)\,|\,0<x<1, 0<t<T\}.$$

It suffices to prove the theorem in Π^T for an arbitrarily fixed $T>0$. Let L_f be a Lipshitz constant of $f_i(x,t,u)$ in u which is uniform in $i\le n$ and $(x,t)\in\overline{\Pi^T}$, L_h be a Lipschitz constant of $h_i(t,v)$ in v which is uniform in $i\le n$ and $t\in[0,T]$.

We split our argument into two claims. In parallel we will derive global a priori estimates, which will be used in the next section.

Claim 1. (6) has a unique continuous solution in $\overline{\Pi^T}$. We first prove that there exists a unique solution $u\in(C(\overline{\Pi^{\theta_0}}))^n$ to (6) for some $\theta_0>0$ such that

$$\omega_i(t;0,\tau) < \omega_j(t;1,\tau) \tag{7}$$

for all

$$\tau\ge 0, t\in[\tau,\tau+\theta_m], k+1\le i\le n, 1\le j\le k,$$

where $m = 0$. For $t \in [0, \theta_0]$ we can express $v(t)$ in the form

$$v_i(t) = \varphi_i(\omega_i(0; x_i, t)) + \int_0^t \Big[u(\omega_i(\tau; x_i, t), \tau) \cdot \int_0^1 \nabla_u f_i(\omega_i(\tau; x_i, t), \tau, \sigma u)\, d\sigma + f_i(\omega_i(\tau; x_i, t), \tau, 0) \Big]\, d\tau, \qquad 1 \le i \le n, \tag{8}$$

where $x_i = 0$ for $1 \le i \le k$ and $x_i = 1$ for $k+1 \le i \le n$.

Convention. In the maximization operators below, unless their range is explicitly specified, we assume the following:

the range of i, x is $i \le n$, $x \in [0,1]$;
the range of i, t is $i \le n$, $t \in [0,T]$;
the range of i, x, t is $i \le n$, $x \in [0,1]$, $t \in [0,T]$;
the range of i, t, z is $i \le n$, $t \in [0,T]$, $\|z\| \le M$;
the range of i, x, t, y is $i \le n$, $x \in [0,1]$, $t \in [0,T]$, $\|y\| \le M$, where a constant M will be specified later.

Apply the contraction mapping principle to (6). Applying the operator defined by the right hand side of (6) to continuous functions u^1 and u^2 and considering the difference $u^1 - u^2$ in $\overline{\Pi^{\theta_0}}$, we get

$$\max_{i \le n; (x,t) \in \overline{\Pi^{\theta_0}}} |u_i^1 - u_i^2| \le \theta_0 q_0 \max_{i \le n; (x,t) \in \overline{\Pi^{\theta_0}}} |u_i^1 - u_i^2|,$$

where

$$q_0 = nL_f(1 + nL_h).$$

Choose

$$\theta_0 = (2q_0)^{-1}.$$

This proves the existence and uniqueness of a $(C(\overline{\Pi^{\theta_0}}))^n$-solution u, satisfying the following local a priori estimate:

$$\max_{i \le n; (x,t) \in \overline{\Pi^{\theta_0}}} |u_i| \le 2\,(1 + nL_h)\,\Phi, \tag{9}$$

where

$$\Phi = \max_{i,x} |\varphi_i(x)| + T \max_{i,x,t} |f_i(x,t,0)| + \max_{i,t} |h_i(t,0)|. \tag{10}$$

Note that the value of q_0 depends on T and does not depend on θ_0. This allows us to complete the proof of the claim in $\lceil T/\theta_0 \rceil$ steps, iterating the local existence-uniqueness result in domains $(\Pi^{j\theta_0} \cap \Pi^T) \setminus \overline{\Pi^{(j-1)\theta_0}}$, where $j \le \lceil T/\theta_0 \rceil$. Simultaneously we arrive at the global a priori estimate

$$\max_{i,x,t} |u_i| \le (3 + 2nL_h)^{\lceil T/\theta_0 \rceil} \Phi. \tag{11}$$

Claim 2. (1)–(3) has a unique C^1-solution in Π^T. We start with a problem for $\partial_x u$:

$$\begin{aligned} \partial_x u_i(x,t) &= (R'_{ix}u)(x,t) \\ &+ \int\limits_{t_i(x,t)}^{t} \Big[\nabla_u f_i(\xi,\tau,u) \cdot \partial_\xi u(\xi,\tau) - \partial_\xi \lambda_i(\xi,\tau)\partial_\xi u_i(\xi,\tau) \\ &+ (\partial_\xi f_i)(\xi,\tau,u)\Big]\Big|_{\xi=\omega_i(\tau;x,t)} d\tau, \qquad 1 \le i \le n, \end{aligned} \tag{12}$$

where

$$(R'_{ix}u)(x,t) = \varphi'_i(\omega_i(0;x,t))$$

if $t_i(x,t) = 0$ and

$$\begin{aligned} (R'_{ix}u)(x,t) = \lambda_i^{-1}(y_i,\tau)\Big[& f_i(y_i,\tau,u) \\ & - \nabla_v h_i(\tau,v) \cdot v'(\tau) - (\partial_t h_i)(\tau,v)\Big]\Big|_{\tau=t_i(x,t)} \end{aligned}$$

otherwise. Here $y_i = 0$ for $k+1 \le i \le n$ and $y_i = 1$ for $1 \le i \le k$. Let us show that there is a unique solution $\partial_x u \in \mathrm{C}\left(\overline{\Pi^{\theta_1}}\right)$ to (12) for some θ_1 satisfying the condition (7) with $m = 1$. Combining (1) with (12) for $t \in [0, \theta_1]$, we get

$$\begin{aligned} v'_i(t) &= f_i(x_i,t,u) - \lambda_i(x_i,t)\partial_x u_i(x_i,t) \\ &= f_i(x_i,t,u) - \lambda_i(x_i,t)\Big[\varphi'_i(\omega_i(0;x_i,t)) \\ &+ \int\limits_0^t \Big[\nabla_u f_i(\xi,\tau,u) \cdot \partial_\xi u(\xi,\tau) - \partial_\xi \lambda_i(\xi,\tau)\partial_\xi u_i(\xi,\tau) \\ &+ (\partial_\xi f_i)(\xi,\tau,u)\Big]\Big|_{\xi=\omega_i(\tau;x_i,t)} d\tau\Big], \quad 1 \le i \le n. \end{aligned} \tag{13}$$

Using the fact that u is a known continuous function (see Claim 1), we now apply the operator defined by the right hand side of (12) to continuous functions $\partial_x u^1$ and $\partial_x u^2$. Notice the estimate

$$\max_{i\le n;(x,t)\in\overline{\Pi^{\theta_1}}} |\partial_x u_i^1 - \partial_x u_i^2| \le \theta_1 q_1 \max_{i\le n;(x,t)\in\overline{\Pi^{\theta_1}}} |\partial_x u_i^1 - \partial_x u_i^2|,$$

where

$$q_1 = \left(nL_f + \max_{i,x,t} |\partial_x \lambda_i|\right)\left(1 + nL_h \max_{i,x,t} |\lambda_i| \max_{i,x,t} |\lambda_i|^{-1}\right).$$

Choose

$$\theta_1 = (2q_1)^{-1}.$$

This shows that the operator defined by the right hand side of (12) has the contraction property with respect to the domain $\overline{\Pi^{\theta_1}}$ and proves the existence and the uniqueness of $u \in \mathrm{C}^{1,0}_{x,t}(\overline{\Pi^{\theta_1}})$. Furthermore,

$$\max_{i\le n;(x,t)\in\overline{\Pi^{\theta_1}}} |\partial_x u_i| \le 2\left(1 + nL_h \max_{i,x,t} |\lambda_i| \max_{i,x,t} |\lambda_i|^{-1}\right)\Psi, \tag{14}$$

where

$$\begin{aligned}\Psi = \max_{i,x} |\varphi_i'| + T \max_{i,x,t,y} |\partial_x f_i| \\ + \max_{i,x,t} |\lambda_i|^{-1} \max_{i,x,t,y} |f_i| + \max_{i,x,t} |\lambda_i|^{-1} \max_{i,t,z} |\partial_t h_i|\end{aligned} \tag{15}$$

and the constant M introduced above in Convention is now set up to

$$M = n\,(3 + 2nL_h)^{\lceil T/\theta_0 \rceil}\,\Phi$$

(see the estimate (11)). Note that q_1 depends on T and does not on θ_1. To complete the proof of the claim, it hence remains to iterate the local existence-uniqueness result in domains $(\Pi^{j\theta_1} \cap \Pi^T) \setminus \overline{\Pi^{(j-1)\theta_1}}$, where $j \le \lceil T/\theta_1 \rceil$. This also gives us the global a priori estimate

$$\max_{i,x,t} |\partial_x u_i| \le \left(3 + 2nL_h \max_{i,x,t} |\lambda_i| \max_{i,x,t} |\lambda_i|^{-1}\right)^{\lceil T/\theta_1 \rceil} \Psi. \tag{16}$$

The fact that u is a C^1-function in both arguments follows now from (1). Furthermore,

$$\max_{i,x,t} |\partial_t u_i| \le \max_{i,x,t,y} |f_i| + \max_{i,x,t} |\lambda_i| \max_{i,x,t} |\partial_x u_i|, \tag{17}$$

where $\partial_x u_i$ satisfy (16). The claim is proved.

Since T is arbitrary, the theorem follows. □

Definition 2.2. *A continuous solution to the integral-operator system (6) is called a continuous solution to the problem (1)–(3).*

From the proof of Claim 1 (in the proof of Theorem 2.1) we obtain also the following fact: If all the initial data in (1)–(3) are continuous functions and f_i and h_i are globally Lipschitz, respectively, in u and v, then there is a unique continuous solution to (1)–(3) satisfying the global a priori estimate (11). This gives us the following continuous dependence theorem.

Theorem 2.3. *Assume that the initial data λ_i, f_i, φ_i, and h_i are continuous functions in their arguments and λ_i are Lipschitz in $x \in [0,1]$. Let $\nabla_y f(x,t,y)$ be bounded on $K \times \mathbb{R}^n$ for every compact $K \subset \overline{\Pi}$ and $\nabla_z h(t,z)$ be bounded on $K \times \mathbb{R}^n$ for every compact $K \subset [0,\infty)$. Suppose that the zero-order compatibility conditions (4) are fulfilled. If $f(x,t,0) \equiv 0$ for all $(x,t) \in \overline{\Pi}$ and $h(t,0) \equiv 0$ for all $t \in [0,\infty)$, then the continuous solution to the problem (1)–(3) continuously depends on $\varphi(x)$.*

3 The Case of Non-Lipschitz Nonlinearities: Existence and Uniqueness Result

We here extend Theorem 2.1 to the case of non-Lipschitz nonlinearities in (1) and (3).

Theorem 3.1. *Assume that the initial data λ_i and f_i are continuous, λ_i and φ_i are C^1-smooth in x, f_i are C^1-smooth in x and u, h_i are C^1-smooth in both arguments. Suppose that for each $T > 0$ there exist $C_f > 0$ and $C_h > 0$ such that*

$$\|\nabla_y f(x,t,y)\| \le C_f \left(\log\log F(x,t,\|y\|)\right)^{1/4}, \tag{18}$$

$$\|\nabla_z h(t,z)\| \le C_h \left(\log\log H(t,\|z\|)\right)^{1/4}, \tag{19}$$

where F (resp., H) is a polynomial in $\|y\|$ (resp., in $\|z\|$) with coefficients in $C^1\left(\overline{\Pi^T}\right)$ (resp., in $C^1[0,T]$). If the zero-order and the first-order compatibility conditions (4) and (5) are fulfilled, then the problem (1)–(3) has a unique classical solution in Π.

Proof. It suffices to prove the theorem in Π^T for an arbitrarily fixed $T > 0$. Let us prove that there exists a unique continuous solution to our problem

(in the sense of Definition 2.2) such that

$$\max_{i,x,t} |u_i| \le e^R/\sqrt{n} \tag{20}$$

for all sufficiently large $R > 0$. On the account of (11), we are done if we show that

$$\Phi\left[3 + 2nC_h\left(\log\log \max_{[0,T]\times[0,e^R]} H(t,\|z\|)\right)^{1/4}\right]^{\lceil T/\theta_0\rceil} \le e^R/\sqrt{n}, \tag{21}$$

where

$$\theta_0 = (2C_f n)^{-1}\left(\log\log \max_{\overline{\Pi^T}\times[0,e^R]} F(x,t,\|y\|)\right)^{-1/4} \times\left(1 + nC_h\left(\log\log \max_{[0,T]\times[0,e^R]} H(t,\|z\|)\right)^{1/4}\right)^{-1}.$$

Let σ be the largest maximum absolute value of coefficients of F and H in $\overline{\Pi^T}$. Let δ be the maximum degree of the polynomials F and H. Set

$$S = \sigma\left(1 + e^R\right)^{\delta}.$$

It is easy to see that

$$\max\left\{\max_{\overline{\Pi^T}\times[0,e^R]} F(x,t,\|y\|), \max_{[0,T]\times[0,e^R]} H(t,\|z\|)\right\} \le S.$$

Obviously, there exists $R_0 > 0$ such that for all $R \ge R_0$ the left hand side of (21) is bounded from above by

$$\Phi\left[\log\log S\right]^{1/2(\log\log S)^{1/2}}.$$

Fix an arbitrary $R \ge R_0$ so that

$$\Phi\left[(1+\delta)\log(2\sigma) + \delta R\right] \le e^R/\sqrt{n}.$$

The desired estimate (21) now follows from the inequality

$$\begin{aligned}
&\Phi\left[\log\log S\right]^{1/2(\log\log S)^{1/2}} \le \\
&\quad \le \Phi \exp\left\{1/2\log\left(\log\log S\right)\left(\log\log S\right)^{1/2}\right\} \\
&\quad = \Phi \exp\left\{\log\left(\log\log S\right)^{1/2}\left(\log\log S\right)^{1/2}\right\} \\
&\quad \le \Phi \exp\left\{\log\log S\right\} = \Phi \log S \le \Phi \log\left(\sigma(1+e^R)^\delta\right) \\
&\quad = \Phi\left[(1+\delta)\log(2\sigma) + \delta R\right] \le e^R/\sqrt{n}.
\end{aligned} \tag{22}$$

The existence and the uniqueness of a continuous solution satisfying the bound (20) is therewith proved. This gives us the unconditional existence and, since $R \ge R_0$ is arbitrary, we have also the unconditional uniqueness.

To prove that the solution is a $\left[\mathrm{C}^{1,0}_{x,t}(\Pi^T)\right]^n$-function, we apply a similar argument, but now use the global a priori estimate (16). It suffices to show that for some $Q > 0$ there is a unique continuous function $\partial_x u$ with

$$\max_{i,x,t} |\partial_x u_i| \le e^Q/\sqrt{n}.$$

Fix $P \ge R_0$ and set up the constant M introduced by Convention in Section 2 to $M = e^P/\sqrt{n}$. Notice the existence of a constant $Q_0 > 0$ such that for all $Q \ge Q_0$ the right hand side of (16) is bounded from above by

$$\Psi\left[\log\log S\right]^{1/2(\log\log S)^{1/2}}$$

and choose $Q \ge Q_0$ satisfying the inequality

$$\Psi\left[(1+\delta)\log(2\sigma) + \delta Q\right] \le e^Q/\sqrt{n}.$$

To finish the proof of the $\left[\mathrm{C}^{1,0}_{x,t}(\Pi^T)\right]^n$-smoothness, it remains to apply the calculation (22) with Ψ in place of Φ.

By (1), u is a $\left[\mathrm{C}^1(\Pi^T)\right]^n$-function. The proof is complete. $\square$

Remark 3.1. To prove the uniqueness part of Theorem 3.1, one can also run a standard argument. Let u and w be two classical solutions to the problem

(1)–(3). Then $u - w$ satisfies the sytem

$$
\begin{aligned}
&(\partial_t + \Lambda(x,t)\partial_x)u = \\
&\qquad = (u-w)\cdot \int_0^1 \nabla_u f\left(x,t,\sigma u + (1-\sigma)w\right)\, d\sigma, \\
&u(x,0) = 0, \\
&u_i(0,t) = (v-\tilde{v})\cdot \int_0^1 \nabla_v h_i\left(x,t,\sigma v + (1-\sigma)\tilde{v}\right)\, d\sigma, \\
&\qquad\qquad\qquad\qquad k+1 \le i \le n; \\
&u_i(1,t) = (v-\tilde{v})\cdot \int_0^1 \nabla_v h_i\left(x,t,\sigma v + (1-\sigma)\tilde{v}\right)\, d\sigma, \\
&\qquad\qquad\qquad\qquad 1 \le i \le k.
\end{aligned}
$$

Here

$$\tilde{v}(t) = (w_1(0,t),\dots,w_k(0,t),w_{k+1}(1,t),\dots,w_n(1,t)).$$

Since $\nabla_u f(x,t,\sigma u + (1-\sigma)w)$ and $\nabla_v h(x,t,\sigma v + (1-\sigma)\tilde{v})$ are known continuous functions, the uniqueness now follows from an analog of (11) for the difference $u - w$.

Remark 3.2. Assume that all conditions excluding (18) and (19) of Theorem 3.1 are fulfilled. Furthermore, assume that $\nabla_z h(t,z)$ is bounded on $K \times \mathbb{R}^n$ for every compact $K \subset [0,\infty)$ and for each $T > 0$ there exists $C_f > 0$ such that

$$\|\nabla_y f(x,t,y)\| \le C_f \log\log F(x,t,\|y\|),$$

where F is a polynomial as in Theorem 3.1. Then, using similar argument as in the proof of Theorem 3.1, one can easily prove that the problem (1)–(3) has a unique classical solution in Π.

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